

Picard Vessiot theory for real partial differential fields

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Abstract

We prove the existence of real Picard-Vessiot extensions for real partial differential fields with real closed field of constants. We establish a Galois correspondence theorem for these Picard-Vessiot extensions and characterize real Liouville extensions of real partial differential fields.

1 Introduction

In [5] and [6], we proved the existence of a real Picard-Vessiot extension for linear differential equations defined over a real ordinary differential field with a real closed field of constants C , we gave an appropriate definition of its differential Galois group, proved that it has the structure of a C -defined linear algebraic group and established a Galois correspondence theorem in this setting. In [4], we gave a characterization of real Liouville extensions of an ordinary differential field in terms of differential Galois groups, which answers a question raised in [8].

In this paper, we establish the analogous results for partial differential fields. The setting up of a Picard-Vessiot theory for real partial differential fields opens the way to the elaboration of a real version of Malgrange's general differential Galois theory (see [2], [10], [11]).

All fields considered will be of characteristic zero. We refer the reader to [3] for the topics on differential Galois theory, to [1] for those on real fields.

2 Existence of real Picard-Vessiot extensions

Following Kolchin [9], for K a field with pairwise commuting derivations $\partial_1, \dots, \partial_m$, we shall denote by K_D the field $K\langle u_1, \dots, u_m \rangle$, where u_1, \dots, u_m

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are differentially algebraically independent over K , endowed with the derivation $D = \sum_{i=1}^m u_i \partial_i$.

Proposition 1. *Let K be a real field endowed with pairwise commuting derivations $\partial_1, \dots, \partial_m$ such that the field of constants C is real closed. Let us be given a differential system*

$$\partial_k Y = A_k Y, \quad 1 \leq k \leq m, \quad (1)$$

where $A_k \in M_{r \times r}(K)$ are such that $\partial_l A_k + A_k A_l = \partial_k A_l + A_l A_k$, $1 \leq k, l \leq m$.

Then there exists a Picard-Vessiot extension of K for (1), which moreover is a real field.

Proof. It is known that there exists a Picard-Vessiot extension L of $K(i)$ for (1), i.e. L is differentially generated over $K(i)$ by the entries of a fundamental matrix $y = (y_{ij})$ of (1) and the field of constants of L is the algebraically closed field $C(i)$.

We consider now the field $K_D := K\langle u_1, \dots, u_m \rangle$, where u_1, \dots, u_m are differentially algebraically independent over K , endowed with the derivation $D = \sum_{i=1}^m u_i \partial_i$. The field of constants of K_D is C (cf. [9] §1). Moreover, K_D is a real field. We consider now the differential system defined over K_D

$$DY = AY, \quad \text{where } A = \sum_{i=1}^m u_i A_i. \quad (2)$$

Clearly a fundamental matrix for (1) with entries in some field extension K' of K is also a fundamental matrix for (2) with entries in the field $K'_D = K'\langle u_1, \dots, u_m \rangle$.

Now, by [5] Corollary 2.6, there exists a real field F which is a Picard-Vessiot extension of K_D for (2). Clearly $F(i)$ is a Picard-Vessiot extension of $K_D(i)$ for (2). On the other hand, as L is generated over $K(i)$ by the entries of a fundamental matrix y for (1), the field $L_D = L\langle u_1, \dots, u_m \rangle$ is generated over $K_D(i)$ by the entries of y , which, by the preceding observation, is as well a fundamental matrix for (2). Hence, L_D is a Picard-Vessiot extension of $K_D(i)$ for (2) and we have a differential $K_D(i)$ -isomorphism from $F(i)$ onto L_D . Let us identify F with its image by this isomorphism. We have then an involution σ in L_D , commuting with the derivation D , such that $\sigma|_F = Id_F$; in particular, $\sigma u_k = u_k$, $1 \leq k \leq m$. This last equality implies that the field L is stable under σ .

Let us consider now the $C(i)$ -vector subspace V of L^r of solutions of (1). As the differential system (1) is defined over K , we have $\sigma(V) \subset V$. Let us write $V^\sigma = \{v \in V : \sigma v = v\}$. Clearly V^σ is a C -vector subspace of V and, if v_1, \dots, v_s is a C -basis of V^σ , it is as well a $C(i)$ -basis of V . Indeed, if $\lambda_1 v_1 + \dots + \lambda_s v_s = 0$, with $\lambda_1, \dots, \lambda_s \in C(i)$, then $\sigma(\lambda_1) v_1 + \dots + \sigma(\lambda_s) v_s = 0$ as well hence we obtain the dependence relations $(\lambda_1 + \sigma(\lambda_1)) v_1 + \dots + (\lambda_s + \sigma(\lambda_s)) v_s = 0$ and $i(\lambda_1 - \sigma(\lambda_1)) v_1 + \dots + i(\lambda_s - \sigma(\lambda_s)) v_s = 0$ with coefficients in C which imply the vanishing of $\lambda_1, \dots, \lambda_s$. Now, a vector v in V may be written as $v = (1/2)(v + \sigma(v)) - (i/2)i(v - \sigma(v))$, where $v + \sigma(v)$ and $i(v - \sigma(v))$ belong to V^σ , hence v can be written as a linear combination of v_1, \dots, v_s with coefficients in $C(i)$. We have then $s = \dim_C V^\sigma = \dim_{C(i)} V = r$. So $L^\sigma = \{x \in L : \sigma x = x\}$ contains a fundamental matrix for (1) and L^σ is a Picard-Vessiot extension of K for (1), which is a real field since $L^\sigma \subset F$.

Remark 2. In their preprint [7], H. Gillet, S. Gorchinskiy and A. Ovchinnikov prove the existence of Picard-Vessiot extensions for real differential fields with real closed field of constants. It is worth pointing out that their method, namely the use of Tannakian categories, does not lead to the existence of a **real** Picard-Vessiot extension.

3 Galois correspondence

Let K be a real partial differential field with real closed field of constants C , $L|K$ a real Picard-Vessiot extension. As in the ordinary case (see [6]), we shall consider the set $DHom_K(L, L(i))$ of K -differential morphisms from L into $L(i)$ and transfer the group structure from $DAut_{K(i)} L(i)$ to $DHom_K(L, L(i))$ by means of the bijection

$$\begin{array}{ccc} DAut_{K(i)} L(i) & \rightarrow & DHom_K(L, L(i)) \\ \tau & \mapsto & \tau|_L \end{array}.$$

The group $DHom_K(L, L(i))$ has the structure of a C -defined (Zariski) closed subgroup of some $C(i)$ -linear algebraic group and we shall denote it as $DGal(L|K)$.

Proposition 3. *The map $DGal(L_D|K_D) \rightarrow DGal(L|K)$ which to each K_D -differential morphism from L_D into $L_D(i)$ assigns its restriction to L is an isomorphism of groups.*

Proof. See [9], Theorem 1.

For a closed subgroup H of $\mathrm{DGal}(L|K)$, L^H is a partial differential subfield of L containing K . If E is an intermediate partial differential field, i.e. $K \subset E \subset L$, then $L|E$ is a real Picard-Vessiot extension and $\mathrm{DGal}(L|E)$ is a C -defined closed subgroup of $\mathrm{DGal}(L|K)$. As in the ordinary case (see [5] Theorem 3.1, [6] Theorem 4.4), we obtain a Galois correspondence theorem.

Theorem 4. *Let K be a real partial differential field with real closed field of constants C , $L|K$ be a real Picard-Vessiot extension, $\mathrm{DGal}(L|K)$ its differential Galois group.*

1. *The correspondences*

$$H \mapsto L^H \quad , \quad E \mapsto \mathrm{DGal}(L|E)$$

define inclusion inverting mutually inverse bijective maps between the set of C -defined closed subgroups H of $\mathrm{DGal}(L|K)$ and the set of partial differential fields E with $K \subset E \subset L$.

2. *The intermediate partial differential field E is a Picard-Vessiot extension of K if and only if the subgroup $\mathrm{DGal}(L|E)$ is normal in $\mathrm{DGal}(L|K)$. In this case, the restriction morphism*

$$\begin{array}{ccc} \mathrm{DGal}(L|K) & \rightarrow & \mathrm{DGal}(E|K) \\ \sigma & \mapsto & \sigma|_E \end{array}$$

induces an isomorphism

$$\mathrm{DGal}(L|K) / \mathrm{DGal}(L|E) \simeq \mathrm{DGal}(E|K).$$

4 Liouville extensions

Definition 5. Let K be a real field endowed with pairwise commuting derivations $\partial_1, \dots, \partial_m$ such that the field of constants C of K is real closed. Let $L|K$ be a partial differential field extension, α an element in L . We say that α is

- an integral over K if $\partial_k \alpha = a_k \in K$, $1 \leq k \leq m$, and a_k is not a derivative in K for all k ;
- the exponential of an integral over K if $\partial_k \alpha / \alpha \in K \setminus \{0\}$, $1 \leq k \leq m$.

From Proposition 3 and the corresponding results in the ordinary case ([4] Examples 7 and 8), we obtain that, if α is an integral over K , then $K\langle\alpha\rangle|K$ is a real Picard-Vessiot extension and its differential Galois group $\mathrm{DGal}(K\langle\alpha\rangle|K)$ is isomorphic to the additive group \mathbb{G}_a ; and, if α is the exponential of an integral and the field $K\langle\alpha\rangle$ is real and with field of constants equal to C , then $K\langle\alpha\rangle|K$ is a Picard-Vessiot extension and $\mathrm{DGal}(K\langle\alpha\rangle|K)$ is isomorphic to the multiplicative group \mathbb{G}_m , or a finite subgroup of it.

Definition 6. A partial differential field extension $K \subset L$ is called a *Liouville extension* (resp. a *generalised Liouville extension*) if there exists a chain of intermediate partial differential fields $K = F_1 \subset F_2 \subset \cdots \subset F_n = L$ such that $F_{i+1} = F_i(\alpha_i)$, where α_i is either an integral or the exponential of an integral over F_i (resp. or α_i is algebraic over F_i).

From the definition, we obtain that $L|K$ is a (generalized) Liouville extension of partial differential fields if and only if the ordinary differential field extension $L_D|K_D$ is a (generalized) Liouville extension.

Definition 7. Let G be a connected solvable linear algebraic group defined over a field C . We say that G is C -split if it has a composition series $G = G_1 \supset G_2 \supset \cdots \supset G_s = 1$ consisting of connected C -defined closed subgroups such that G_i/G_{i+1} is C -isomorphic to \mathbb{G}_a or \mathbb{G}_m , $1 \leq i < s$.

From the results obtained in the ordinary case ([4] Section 3, Theorems 17 and 18), we obtain the characterization of real Liouville extensions of real partial differential fields.

Theorem 8. Let K be a real partial differential field with real closed field of constants C , $L|K$ be a real Picard-Vessiot extension, $\mathrm{DGal}(L|K)$ be its differential Galois group. The following conditions are equivalent.

1. $L|K$ is a generalized Liouville extension.
2. L is contained in a generalized Liouville extension of K .
3. The identity component of $\mathrm{DGal}(L|K)$ is solvable and C -split.

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